

NEW INEQUALITIES FOR QUANTUM VON NEUMANN AND TOMOGRAPHIC MUTUAL INFORMATION

V.I. Man'ko¹ and L.A. Markovich^{2*}

¹*P.N. Lebedev Physical Institute, Russian Academy of Sciences
Leninskii Prospect 53, Moscow 119991, Russia*

²*Institute of Control Sciences, Russian Academy of Sciences
Profsoyuznaya 65, Moscow 117997, Russia*

*Corresponding author e-mail: kimo1 @ mail.ru

Abstract

Entropic inequalities related to the quantum mutual information for bipartite system and tomographic mutual information is studied for Werner state of two qubits. Quantum correlations corresponding to entanglement properties of the qubits in Werner state are discussed.

Keywords: Entropic inequalities, quantum information, Werner state, tomographic probabilities, qudit.

1 Introduction

The two-qubit systems can demonstrate quantum correlations and these correlations correspond to entanglement phenomenon [1] or to the violation of Bell inequalities [2]. Also the correlations can be associated with quantum discord [3, 4]. The quantum discord is related to difference of classical Shannon information behavior [5] and quantum information behavior determined by von Neumann entropy of a composite bipartite systems and the entropies of its subsystems. Recently [6, 7] the tomographic probability representation of spin (qudit) states was introduced. In this representation the qudit states are identified with the spin-tomogram which is fair probability distribution function determined by the density operator of the states. The relation of the density operator to the spin-tomogram is invertible. Due to this the tomogram contains the complete information on the qudit state. For several qudits the spin-tomogram is also determined as the state density operator and it is a joint probability distribution which provides the possibility to reconstruct the density operator. Since the qudit state in the tomographic probability representation is identified with the standard probability distribution one can use all the characteristics of the distributions like Shannon entropy and information as well as other entropies [8, 9]. The von Neumann entropy was shown [11] to be the minimum of the spin-tomographic Shannon entropy with respect to all the unitary transforms in Hilbert space of the qudit system. There exist different kinds of entropic inequalities for both classical and quantum systems [12–16]. The inequalities relating spin-tomographic and von Neumann entropies were used both for composite and noncomposite systems in [17–20]. The particular quantum state which has properties to be either separable or entangled depending on the parameter values of its density matrix is the Werner state [21] of two qubits.

The aim of our work is to study the tomographic Shannon and von Neumann entropies and informations discussed [20] on the example of the Werner state. We discuss the quantum correlations in the state using the specific characteristics of two-qubit density matrix. This characteristics is the difference of quantum von Neumann information and maximum of the Shannon tomographic information taken with respect to all the local unitary transforms in the Hilbert space of this bipartite qubit systems. We calculate explicitly the characteristics and analyze this parameter behavior as function of Werner state parameters.

The paper is organized as follows. In Sec. 2 we review the tomographic probability representation example of Werner state and introduce the tomographic Shannon information and entropy for this two-qubit state. In Sec. 3 we discuss the maximum of the spin-tomographic entropy of the composite two-qubit system with respect to the local unitary transforms in the Hilbert space.

2 Entropy and information for the Werner state

The tomographic probability distribution for spin states provides the possibility to describe the states with density matrix ρ of two qubits by means of tomogram. By definition the spin tomogram is

$$\omega(m_1, m_2, \bar{n}_1, \bar{n}_2) = \langle m_1, m_2 | U \cdot \rho \cdot U^\dagger | m_1, m_2 \rangle. \quad (1)$$

Here $m_{1,2} = -j, -j+1, \dots, j$, $j = 0, 1/2, 1 \dots$ are spin projections and U is the rotation matrix

$$U = \begin{pmatrix} \cos \frac{\theta_1}{2} e^{\frac{i(\varphi_1 + \psi_1)}{2}} & \sin \frac{\theta_1}{2} e^{\frac{i(\varphi_1 - \psi_1)}{2}} \\ -\sin \frac{\theta_1}{2} e^{\frac{i(\psi_1 - \varphi_1)}{2}} & \cos \frac{\theta_1}{2} e^{\frac{-i(\varphi_1 + \psi_1)}{2}} \end{pmatrix} \otimes \begin{pmatrix} \cos \frac{\theta_2}{2} e^{\frac{i(\varphi_2 + \psi_2)}{2}} & \sin \frac{\theta_2}{2} e^{\frac{i(\varphi_2 - \psi_2)}{2}} \\ -\sin \frac{\theta_2}{2} e^{\frac{i(\psi_2 - \varphi_2)}{2}} & \cos \frac{\theta_2}{2} e^{\frac{-i(\varphi_2 + \psi_2)}{2}} \end{pmatrix}. \quad (2)$$

The matrix (2) is considered as the direct product of two matrices of irreducible representations of $SU(2)$ - group [10]. The Werner state of two qubits is determined by density matrix [21] of the form

$$\rho_W(p) = \begin{pmatrix} \rho_{1111} & \rho_{1112} & \rho_{1121} & \rho_{1122} \\ \rho_{1211} & \rho_{1212} & \rho_{1221} & \rho_{1222} \\ \rho_{2111} & \rho_{2112} & \rho_{2121} & \rho_{2122} \\ \rho_{2211} & \rho_{2212} & \rho_{2221} & \rho_{2222} \end{pmatrix} = \begin{pmatrix} \frac{1+p}{4} & 0 & 0 & \frac{p}{2} \\ 0 & \frac{1-p}{4} & 0 & 0 \\ 0 & 0 & \frac{1-p}{4} & 0 \\ \frac{p}{2} & 0 & 0 & \frac{1+p}{4} \end{pmatrix}, \quad (3)$$

where parameter $-\frac{1}{3} \leq p \leq 1$. The parameter domain $\frac{1}{3} < p \leq 1$ corresponds to the entangled state.

The eigenvalues of (3) are

$$\lambda_1 = \frac{1+3p}{4}, \quad \lambda_{2,3,4} = \frac{1-p}{4}.$$

The reduced density matrices of the first and the second qubit are the following

$$\begin{aligned} \rho_1 &= \begin{pmatrix} \rho_{1111} + \rho_{1212} & \rho_{1121} + \rho_{1222} \\ \rho_{2111} + \rho_{2212} & \rho_{2121} + \rho_{2222} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ \rho_2 &= \begin{pmatrix} \rho_{1111} + \rho_{2121} & \rho_{1112} + \rho_{2122} \\ \rho_{1211} + \rho_{2221} & \rho_{1212} + \rho_{2222} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Hence the von Neumann entropies of both qubit states and the entropy of the whole system are

$$\begin{aligned} S_1 &= -Tr \rho_1 \ln \rho_1 = \ln 2, & S_2 &= -Tr \rho_2 \ln \rho_2 = \ln 2, \\ S_{12} &= -Tr \rho(p) \ln \rho(p) = -\frac{1+3p}{4} \ln \left(\frac{1+3p}{4} \right) - 3\frac{1-p}{4} \ln \left(\frac{1-p}{4} \right). \end{aligned} \quad (4)$$

The quantum information is defined as the difference of the sum of the entropies of the first and the second qubit states and the entropy of the two-qubit state, i.e.

$$I_q = S_1 + S_2 - S_{12}. \quad (5)$$

Obviously, the quantum information satisfies the inequality $I_q \geq 0$. To construct the state tomogram we have to calculate the diagonal matrix elements of the density matrix in unitary rotated basis in system Hilbert space.

The diagonal matrix elements of the matrix $U \cdot \rho \cdot U^\dagger$ are the following

$$\begin{aligned} \omega_{11}(\uparrow, \uparrow) &= \frac{1}{4} (p (\cos \theta_1 \cos \theta_2 + \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2) + 1), \\ \omega_{22}(\uparrow, \downarrow) &= \frac{1}{4} (1 - p (\cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2)), \\ \omega_{33}(\downarrow, \uparrow) &= \frac{1}{4} (1 - p (\cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2)), \\ \omega_{44}(\downarrow, \downarrow) &= \frac{1}{4} (p (\cos \theta_1 \cos \theta_2 + \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2) + 1). \end{aligned} \quad (6)$$

Above we introduced the notations for the tomographic probabilities given by equation (1), for example $\omega_{11}(\uparrow, \uparrow) \equiv \omega(+\frac{1}{2}, +\frac{1}{2}, \bar{n}_1, \bar{n}_2)$. It is easy to verify that the trace of the rotated density matrix satisfies the normalization condition

$$Tr(U \cdot \rho \cdot U^\dagger) = \omega_{11}(\uparrow, \uparrow) + \omega_{22}(\uparrow, \downarrow) + \omega_{33}(\downarrow, \uparrow) + \omega_{44}(\downarrow, \downarrow) = 1.$$

Marginal distributions corresponding to the first and the second qubit density matrix are

$$\begin{aligned} W_1(\uparrow, \bar{n}_1) &= \omega_{11}(\uparrow, \uparrow) + \omega_{22}(\uparrow, \downarrow), & W_1(\downarrow, \bar{n}_1) &= \omega_{33}(\downarrow, \uparrow) + \omega_{44}(\downarrow, \downarrow), \\ W_2(\uparrow, \bar{n}_2) &= \omega_{11}(\uparrow, \uparrow) + \omega_{33}(\downarrow, \uparrow), & W_2(\downarrow, \bar{n}_2) &= \omega_{22}(\uparrow, \downarrow) + \omega_{44}(\downarrow, \downarrow). \end{aligned}$$

Thus, according to definition of Shannon entropy [5] we can construct the following tomographic entropies of the qubit subsystems

$$\begin{aligned} H_1 &= -W_1(\uparrow, \bar{n}_1) \ln W_1(\uparrow, \bar{n}_1) - W_1(\downarrow, \bar{n}_1) \ln W_1(\downarrow, \bar{n}_1) = \ln 2, \\ H_2 &= -W_2(\uparrow, \bar{n}_2) \ln W_2(\uparrow, \bar{n}_2) - W_2(\downarrow, \bar{n}_2) \ln W_2(\downarrow, \bar{n}_2) = \ln 2. \end{aligned} \quad (7)$$

The tomographic Shannon entropy of the bipartite system reads

$$H_{12} = -\omega_{11}(\uparrow, \uparrow) \ln \omega_{11}(\uparrow, \uparrow) - \omega_{22}(\uparrow, \downarrow) \ln \omega_{22}(\uparrow, \downarrow) - \omega_{33}(\downarrow, \uparrow) \ln \omega_{33}(\downarrow, \uparrow) - \omega_{44}(\downarrow, \downarrow) \ln \omega_{44}(\downarrow, \downarrow). \quad (8)$$

We define the information I_t as maximum of the sum of the difference between the sum of entropies (7) of subsystems and the entropy of the whole system (8)

$$I_t = \max_{\psi_1, \psi_2, \theta_1, \theta_2} (H_1 + H_2 - H_{12}) \quad (9)$$

and it satisfies the inequality $I_t \geq 0$.

3 Maximum of the Shannon information

Let us introduce the following notation $\tilde{H} \equiv \tilde{H}(\psi_1, \psi_2, \theta_1, \theta_2, p) = H_1 + H_2 - H_{12}$. Using (7), (8) and (6) it is straightforward to verify that

$$\begin{aligned} \tilde{H} &= \ln 4 - \frac{1}{2} \ln \left(\frac{1}{4} (1 - p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 - p \cos \theta_1 \cos \theta_2) \right) \\ &\cdot (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 - 1) \\ &+ \frac{1}{2} \ln \left(\frac{1}{4} (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + 1) \right) \\ &\cdot (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + 1). \end{aligned} \quad (10)$$

To find the maximum of \tilde{H} with respect to angles $\psi_1, \psi_2, \theta_1, \theta_2$ we must first find its stationary points. Hence, taking the first derivatives

$$\begin{aligned} \frac{\partial(\tilde{H})}{\partial \theta_1} &= \frac{p}{2} (\cos \theta_2 \sin \theta_1 - \cos(\psi_1 + \psi_2) \cos \theta_1 \sin \theta_2) \\ &\cdot \left(\ln \left(\frac{1}{4} (1 - p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 - p \cos \theta_1 \cos \theta_2) \right) \right. \\ &\left. - \ln \left(\frac{1}{4} (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + 1) \right) \right), \\ \frac{\partial(\tilde{H})}{\partial \theta_2} &= \frac{p}{2} (\cos \theta_1 \sin \theta_2 - \cos(\psi_1 + \psi_2) \cos \theta_2 \sin \theta_1) \\ &\cdot \left(\ln \left(\frac{1}{4} (1 - p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 - p \cos \theta_1 \cos \theta_2) \right) \right. \\ &\left. - \ln \left(\frac{1}{4} (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + 1) \right) \right), \\ \frac{\partial(\tilde{H})}{\partial \psi_1} &= \frac{\partial(\tilde{H})}{\partial \psi_2} = \frac{p}{2} \sin(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 \\ &\cdot \left(\ln \left(\frac{1}{4} (1 - p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 - p \cos \theta_1 \cos \theta_2) \right) \right. \\ &\left. - \ln \left(\frac{1}{4} (p \cos \theta_1 \cos \theta_2 + p \cos(\psi_1 + \psi_2) \sin \theta_1 \sin \theta_2 + 1) \right) \right) \end{aligned}$$

and equating them to zero we can obtain that the critical points $\Theta^0 = (\theta_1^0, \theta_2^0, \psi_1^0, \psi_2^0)$ are

- $\theta_1 = \theta_2 = \pi n, n = 0, 1, \dots$ for $\forall \psi_1, \psi_2$,
- $\theta_1 = \pi/2 + \pi n, \theta_2 = \pi n, n = 0, 1, \dots$ for $\forall \psi_1, \psi_2$,
- $\theta_1 = \pi n, \theta_2 = \pi/2 + \pi n, n = 0, 1, \dots$ for $\forall \psi_1, \psi_2$,

- $\theta_1 = \theta_2 = \pi/2 + \pi n, n = 0, 1, \dots$ for $\psi_1 + \psi_2 = \pi m$ or $\psi_1 + \psi_2 = \pi/2 + \pi m, m = 0, 1, \dots$,
- $\theta_1 = \pi/2 + \pi n, \psi_1 + \psi_2 = \pi/2 + \pi m$ for $\forall \theta_2, n, m = 0, 1, \dots$,
- $\theta_2 = \pi/2 + \pi n, \psi_1 + \psi_2 = \pi/2 + \pi m$ for $\forall \theta_1, n, m = 0, 1, \dots$

The second differential can be written in a quadratic form $d^2\tilde{H}(\Theta)$ with determinant

$$\begin{vmatrix} \frac{\partial^2(\tilde{H})}{\partial\theta_1^2} & \frac{\partial^2(\tilde{H})}{\partial\theta_2\partial\theta_1} & \frac{\partial^2(\tilde{H})}{\partial\psi_1\partial\theta_1} \\ \frac{\partial^2(\tilde{H})}{\partial\theta_1\partial\theta_2} & \frac{\partial^2(\tilde{H})}{\partial\theta_2^2} & \frac{\partial^2(\tilde{H})}{\partial\psi_1\partial\theta_2} \\ 2\frac{\partial^2(\tilde{H})}{\partial\theta_1\partial\psi_1} & 2\frac{\partial^2(\tilde{H})}{\partial\theta_2\partial\psi_1} & 2\frac{\partial^2(\tilde{H})}{\partial\psi_1^2} \end{vmatrix}, \quad (11)$$

where we noticed that $\frac{\partial^2(\tilde{H})}{\partial\theta\partial\psi_1} = \frac{\partial^2(\tilde{H})}{\partial\theta\partial\psi_2}$. According to a sufficient condition for an extremum if $d^2\tilde{H}(\Theta^0)$ is negatively defined quadratic form then Θ^0 is a strict maximum of the function $\tilde{H}(\psi_1, \psi_2, \theta_1, \theta_2, p)$. By Sylvester's criterion, if all of the leading principal minors of (11) are negative, then the quadratic form $d^2\tilde{H}(\Theta^0)$ is negative.

For example we can take $\theta_1 = \theta_2 = \pi/2 + \pi n, n = 0, 1, \dots$. Then determinant (11) for $\psi_1 + \psi_2 = \pi m, m = 0, 1, \dots$ is

$$\begin{vmatrix} \frac{p}{2} \left(\ln \left(\frac{1-p}{4} \right) - \ln \left(\frac{1+p}{4} \right) \right) & \frac{p}{2} \left(\ln \left(\frac{1-p}{4} \right) - \ln \left(\frac{1+p}{4} \right) \right) & 0 \\ \frac{p}{2} \left(\ln \left(\frac{1-p}{4} \right) - \ln \left(\frac{1+p}{4} \right) \right) & \frac{p}{2} \left(\ln \left(\frac{1-p}{4} \right) - \ln \left(\frac{1+p}{4} \right) \right) & 0 \\ 0 & 0 & p \left(\ln \left(\frac{1-p}{4} \right) - \ln \left(\frac{1+p}{4} \right) \right) \end{vmatrix},$$

and for $\psi_1 + \psi_2 = \pi/2 + \pi m$ it is

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2p^2 \end{vmatrix}$$

It is straightforward to verify that both determinants are equal to zero. Thus, $\Theta_1^0 = (\pi/2 + \pi n, \pi/2 + \pi n, \psi_1 + \psi_2 = \pi m$ or $\psi_1 + \psi_2 = \pi/2 + \pi m, n, m = 0, 1, \dots$ is not an extremum point. Similarly, for all other stationary points it can be proved that the second differential (11) becomes zero. Hence there is no global extremum of the function $\tilde{H}(\psi_1, \psi_2, \theta_1, \theta_2, p)$.

Due to the form of stationary points it is clear that we can find θ_1^0, θ_2^0 that maximize $\tilde{H}(\psi_1, \psi_2, \theta_1, \theta_2, p)$ with fixed angles $\psi_1 + \psi_2 = \pi m$ or $\psi_1 + \psi_2 = \pi/2 + \pi m, m = 0, 1, \dots$

The difference of quantum information I_q and maximum of the unitary tomographic information I_t is

$$I_q - I_t = \Delta I \geq 0. \quad (12)$$

For fixed angles ψ_1, ψ_2 this difference is shown in Figure 1 for $p = 0.9$ and in Figure 2 for $p = 0.999$. It is visible that with increasing of parameter p the minimal value of difference (12) increases. In Figure 1 the minimal value of (12) is about 0.65 and in Figure 2 is about 0.7. Let us find its limit value as $p \rightarrow 1$.

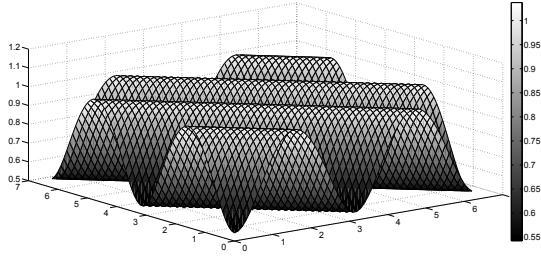


Figure 1: Difference (12) for fixed angles ψ_1, ψ_2 and $p = 0.9$

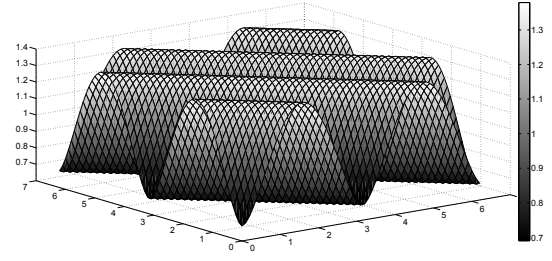


Figure 2: Difference (12) for fixed angles ψ_1, ψ_2 and $p = 0.999$

To this end we use the well-known relation $\lim_{x \rightarrow 0} x \ln x = 0$. Hence, we obtain that (4) and (5) are

$$\begin{aligned} \lim_{p \rightarrow 1} S_{12} &= 0, & \lim_{p \rightarrow -1/3} S_{12} &= -\ln 3 \approx -1.098612, \\ \lim_{p \rightarrow 1} I_q &= \ln 4, & \lim_{p \rightarrow -1/3} I_q &= \ln 4 - \ln 3 \approx 0.287682. \end{aligned}$$

For $\psi_1 + \psi_2 = \pi/2 + \pi m$ and $(\theta_1, \theta_2) = (\pi, \pi)$ the Shannon entropy (10) can be rewritten as

$$\tilde{H}(p) = \ln 4 - \ln(1/4 - p/4)(p/2 - 1/2) + \ln(p/4 + 1/4)(p/2 + 1/2)$$

Then its limits are

$$\lim_{p \rightarrow 1} \tilde{H}(p) = \ln 4 - \ln 2, \quad \lim_{p \rightarrow -1/3} \tilde{H}(p) = \frac{5}{3} \ln 2 + \ln 3.$$

Hence the limit values of (12) are

$$\lim_{p \rightarrow 1} (I_q - I_t) = \ln 2 \approx 0.693147, \quad \lim_{p \rightarrow -1/3} (I_q - I_t) = \frac{1}{3} \ln 2 \approx 0.231049.$$

For $\psi_1 + \psi_2 = \pi/2 + \pi m$ and $(\theta_1, \theta_2) = (\pi, \pi/2)$ entropy (10) is

$$\lim_{p \rightarrow 1} \tilde{H}(p) = 0, \quad \lim_{p \rightarrow -1/3} \tilde{H}(p) = 0$$

and the limit values of (12) are

$$\lim_{p \rightarrow 1} (I_q - I_t) = \ln 4 \approx 1.386294, \quad \lim_{p \rightarrow -1/3} (I_q - I_t) = \ln 4 - \ln 3 \approx 0.287682.$$

These limits can be seen in Figure 3 in all the stationary points with a varying p and additionally for the varying angle $\theta_1 \in [0, 2\pi]$ in Figure 4. Hence, the minimum value of (12) is $\Delta I = \ln 2$ as $p \rightarrow 1$ and $\Delta I = \frac{1}{3} \ln 2$ as $p \rightarrow -\frac{1}{3}$.

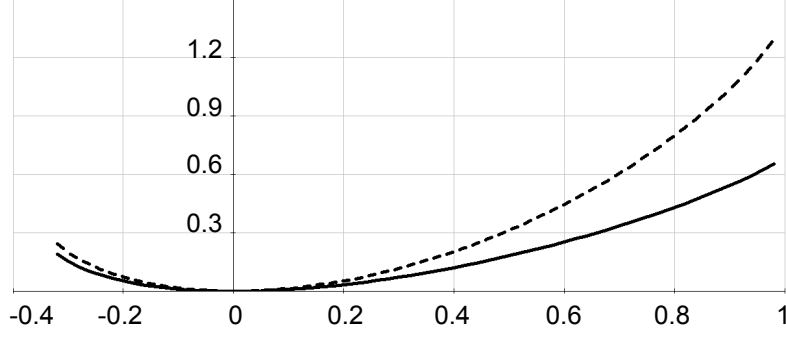


Figure 3: $I_q - I_t$ Solid line: $(\theta_1 = \pi, \theta_2 = \pi, \psi_1 + \psi_2 = \pi m)$, $(\theta_1 = \pi/2, \theta_2 = \pi/2, \psi_1 + \psi_2 = \pi m)$, $(\theta_1 = \pi, \theta_2 = \pi, \psi_1 + \psi_2 = \pi/2 + \pi m)$. Dashed line: $(\theta_1 = \pi/2, \theta_2 = \pi, \psi_1 + \psi_2 = \pi m)$, $(\theta_1 = \pi, \theta_2 = \pi/2, \psi_1 + \psi_2 = \pi m)$, $(\theta_1 = \pi/2, \theta_2 = \pi, \psi_1 + \psi_2 = \pi/2 + \pi m)$, $(\theta_1 = \pi, \theta_2 = \pi/2, \psi_1 + \psi_2 = \pi/2 + \pi m)$, $(\theta_1 = \pi/2, \theta_2 = \pi/2, \psi_1 + \psi_2 = \pi/2 + \pi m)$

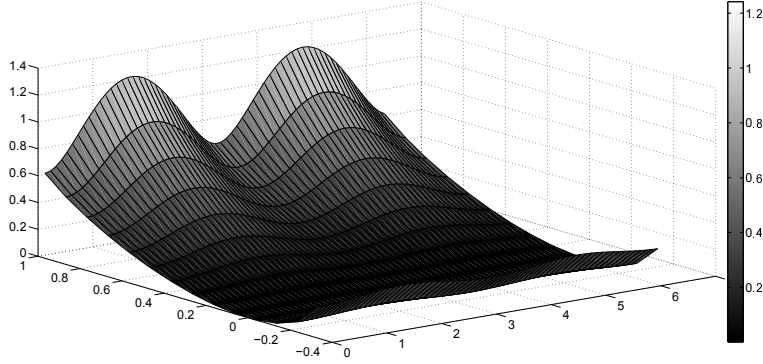


Figure 4: $I_q - I_t$ for $\psi_1 + \psi_2 = \pi$, $-1/3 < p < 1$ for $\theta_1 \in [0, 2\pi]$

4 Summary

To conclude we point out the main results of the work. We studied the correlations in Werner state of two qubits. The difference of von Neumann information I_q and the maximum of tomographic information I_t associated with correlations in the system must be nonnegative. This is shown in Figure 4 for a fixed $\bar{n}_2 = (\theta_2, \psi_2)$ and varying latitude of \bar{n}_1 . For $p \rightarrow 1$ (maximally entanglement state) the difference tends to $\ln 2$. The studied difference characterizes the degree of quantum correlations in the two-qubit system.

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References

- [1] E. Schrödinger, *Die gegenwertige Situation in der Quantenmechanik (The present situation in quantum mechanics)*, *Naturwissenschaften*, **23**, 807–812; 823–828; 844–849 (1935).
- [2] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, *Proposed experiment to test local hidden-variable theories*, *Phys. Rev. Lett.*, **23** (15), 880–4 (1969).
- [3] V.I. Man’ko, A. Yurkevich, *Tomographic Discord and Quantum Correlations in a System of Qubits*, *J. Russ. Laser Res.*, **34** (5), 463–467 (2013).
- [4] K. Modi, Aharon Brodutch, H. Cable, T. Paterek, and V. Vedral, *Quantum discord and other measures of quantum correlation*, *Rev. Mod. Phys.*, **84**, 1655–1707 (2012).
- [5] C. E. Shannon, *A Mathematical Theory of Communication*, *Bell System Technical Journal*, **27**, 379 (1948).
- [6] V.V. Dodonov, V.I. Man’ko, *Positive distribution description for spin states*, *Phys. Lett. A*, **229**, 335–339 (1997).
- [7] V.I. Man’ko, O.V. Man’ko, *Spin state tomography*, *JETP*, **85** (3), 430 (1997).
- [8] A. Rényi, *On measures of information and entropy*, *Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability*, 547–561 (1961).
- [9] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, *Journal of Statistical Physics*, **52**, 479–487 (1988).
- [10] V.I. Man’ko and L.A. Markovich, *Entropic inequalities and properties of some special functions*, *J. Russ. Laser Res.*, **35** (2), (2014).
- [11] R. Vilela Mendes, V.I. Man’ko, *On the problem of quantum control in infinite dimensions*, *Journal of Physics A: Math. Theor.*, **44** (13), 135302 (2011).
- [12] E. H. Lieb, M. B. Ruskai, *Proof of the Strong Subadditivity of Quantum Mechanical Entropy*, *J. Math. Phys.*, **14**, 1938–1941, (1973).
- [13] S. Wehner, A. Winter, *Entropic uncertainty relationsa survey*, *New J. Phys.*, **12**, 025009 (2010).
- [14] E.H. Lieb, M.B. Ruskai, *Some Operator Inequalities of the Schwarz Type*, *Adv. Math.*, **12**, 269–273 (1974).
- [15] M. Ohya, D. Petz, *Quantum entropy and its use*, Springer-Verlag, Berlin, (1993).

- [16] Alexey E. Rastegin, *Fano type quantum inequalities in terms of q -entropies*, **11** (6), 1895–1910, (2012).
- [17] V.N. Chernega, O.V. Man'ko, V.I. Man'ko, *Generalized qutrit portrait of the qutritstate density matrix*, *J. Russ. Laser Res.*, **34** (4), 383–387 (2013).
- [18] V.N. Chernega, O.V. Man'ko, *Tomographic and improved subadditivity conditions for two qubits and qudit with $j = 3/2$* , *J. Russ. Laser Res.*, **35** (1), 27–38 (2014).
- [19] M.A. Man'ko, V.I. Man'ko, *Quantum correlations expressed as information and entropic inequalities for composite and noncomposite systems*, *J. Phys. Conf. Ser. in the Proceedings of the XVI Symposium "Symmetries in Science"*, (Bregenz, Austria, July 21–26, 2013)
- [20] V.N. Chernega, O.V. Man'ko, V.I. Man'ko, *Subadditivity condition for spin-tomograms and density matrices of arbitrary composite and noncomposite qudit systems*, <http://arxiv.org/abs/1403.2233> (2014).
- [21] R.F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, *Phys. Rev. A*, **40**, 4277 (1989).
- [22] M.A. Man'ko and V.I. Man'ko, arXiv:1312.6988 (2013), to appear in *Physica Scripta* (2014).